Introduction to Combinatorial Anabelian Geometry III

$\boxed{\S 1}$

K: an NF or an MLF $\hookrightarrow$ $\overline{K}$: an alg closure of K

$G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$

(g, r): a pair of nonnegative integers s.t. $2g - 2 + r > 0$

C: a hyperbolic curve of type (g, r) / K
(g: the genus, r: the number of cusps)

$\pi_1((-))$: the étale fundamental gp of $(-)$

<u>Recall</u>: The homotopy ext seq

$$1 \to \pi_1(C \times_K \overline{K}) \to \pi_1(C) \to G_K \to 1$$

induces an outer representation

$$\rho : G_K \to \mathrm{Out}(\pi_1(C \times_K \overline{K}))$$

<u>Belyĭ, Voevodskiĭ, Matsumoto</u>

If $r > 0$, then ρ is injective. ■

Today,

$\boxed{\text{Thm 1}}$ (Hoshi-Mochizuki)

For any (g, r), ρ is injective. ■

<u>Method</u>: Combinatorial Anabelian Geometry

Notations:

k: an alg closed field of char 0

X: a hyperbolic curve of type (g, r) / k

$X_n \overset{\text{def}}{=} \{(x_1, \cdots, x_n) \in \overbrace{X \times_k \cdots \times_k X}^{n} \mid x_i \neq x_j \text{ if } i \neq j\}$

$\Pi_n \overset{\text{def}}{=} \pi_1(X_n)$

In particular, the projections

$$\begin{array}{ccccccccc} X_n & \to & X_{n-1} & \to & \cdots & \to & X_2 & \to & X \\ (x_1, \cdots, x_n) & \mapsto & (x_1, \cdots, x_{n-1}) & \mapsto & \cdots & \mapsto & (x_1, x_2) & \mapsto & x_1 \end{array}$$

induce a standard sequence of [outer] surjections

$$\Pi_n \twoheadrightarrow \Pi_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1$$

Write $K_m \overset{\text{def}}{=} \mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_m)$, $\Pi_0 \overset{\text{def}}{=} \{1\}$.

$\rightsquigarrow \{1\} = K_n \subseteq K_{n-1} \subseteq \cdots \subseteq K_1 \subseteq K_0 = \Pi_n$

$\alpha \in \mathrm{Aut}(\Pi_n)$ is F-admissible $\overset{\text{def}}{\Leftrightarrow}$ For any fiber subgroup $J \subseteq \Pi_n$, it holds that $\alpha(J) = J$.

Recall:

$J \subseteq \Pi_n$ is a fiber subgroup $\overset{\text{def}}{\Leftrightarrow}$ $J = \mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_{n'})$
— where $\Pi_n \twoheadrightarrow \Pi_{n'}$ is induced by a proj $X_n \to X_{n'}$.

$\alpha \in \mathrm{Aut}(\Pi_n)$ is C-admissible $\overset{\text{def}}{\Leftrightarrow}$

(i) $\alpha(K_m) = K_m \quad (0 \leq m \leq n)$

(ii) $\alpha : K_m/K_{m+1} \xrightarrow{\sim} K_m/K_{m+1}$ induces a bijection between the set of cusp'l inertia subgps $\subseteq K_m/K_{m+1}$

$\alpha \in \mathrm{Aut}(\Pi_n)$ is FC-admissible $\overset{\text{def}}{\Leftrightarrow}$ α is F-admissible and C-admissible

$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n) \overset{\text{def}}{=} \{$ FC-admissible automorphisms of Π_n $\}$

$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \overset{\text{def}}{=} \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)/\mathrm{Inn}(\Pi_n)$

<u>Observe</u>: $X_{n+1} \to X_n$ "forgetting the last factor" induces

$$\phi_n : \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \ \to \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

$\boxed{\text{Thm 2}}$ ϕ_n is injective for $n \geq 1$. ■

<u>Remark</u>:

(i) ϕ_n is bijective for $n \geq 4$.

(ii) Various $X_{n+1} \to X_n$ "forgetting a factor" induce the same $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$.

$\boxed{\text{Thm } 2} \Rightarrow \boxed{\text{Thm } 1}$

Let $X \stackrel{\text{def}}{=} C \times_K \overline{K}$, $k \stackrel{\text{def}}{=} \overline{K}$

($\rightsquigarrow \pi_1(C \times_K \overline{K}) = \pi_1(X) = \Pi_1$)

<u>Note</u>: The outer rep'n $\rho : G_K \to \mathrm{Out}(\Pi_1)$ factors as

$$G_K \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1) \hookrightarrow \mathrm{Out}(\Pi_1).$$

Thus, to show that ρ is injective, it suffices to show that

$$G_K \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1) \text{ is injective.}$$

This follows from the commutativity of the diagram

$$\begin{tikzcd}
G_K \arrow[r] \arrow[d, equal] & \mathrm{Out}^{\mathrm{FC}}(\Pi_1) & \\
G_K \arrow[r] \arrow[d, equal] & \mathrm{Out}^{\mathrm{FC}}(\Pi_2) \arrow[u, hook, "\phi_1"'] & \\
G_K \arrow[r] \arrow[rr, bend right, "\text{injective by Bely\u{\i}}"'] & \mathrm{Out}^{\mathrm{FC}}(\Pi_3) \arrow[u, hook, "\phi_2"'] \arrow[r, dashed, "\exists"] & \mathrm{Out}(\pi_1(\mathbb{P}^1_{\overline{K}} \setminus \{0, 1, \infty\}))
\end{tikzcd}$$

Today, for simplicity, we consider the proof of the injectivity of $\phi_1 : \mathrm{Out}^{\mathrm{FC}}(\Pi_2) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$.

$\Rightarrow$ It suffices to verify:

$\boxed{\text{Prop 3}}$

Let

$$\mathrm{Aut}^{\mathrm{IFC}}(\Pi_2) \overset{\mathrm{def}}{=} \left\{ \alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2) \;\middle|\; \begin{array}{ccl} & & \Pi_1 \curvearrowleft \alpha_1 = \mathrm{id} \\ & \overset{p_1}{\nearrow\!\!\!\!\nearrow} & \\ \alpha \curvearrowright \Pi_2 & & \\ & \underset{p_2}{\searrow\!\!\!\!\searrow} & \\ & & \Pi_1 \curvearrowleft \alpha_2 = \mathrm{id} \end{array} \right\},$$

$$\Xi \overset{\mathrm{def}}{=} \mathrm{Ker}(p_1) \cap \mathrm{Ker}(p_2) \quad [\subseteq \Pi_2].$$

Then the injection

$$\Xi \overset{\mathrm{conj.}}{\hookrightarrow} \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2) \quad [\text{cf. the slimness of } \Pi_2]$$

is bijective. $\blacksquare$

Indeed, let $\sigma \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$ s.t.

$$\begin{array}{ccl} & & \Pi_1 \curvearrowleft \sigma_1 = \mathrm{Inn}(g_1) \\ & \overset{p_1}{\nearrow\!\!\!\!\nearrow} & \\ \sigma \curvearrowright \Pi_2 & & \\ & \underset{p_2}{\searrow\!\!\!\!\searrow} & \\ & & \Pi_1 \curvearrowleft \sigma_2 \end{array}$$

Observe: Let $\mathbb{D} \subseteq \Pi_2$ be a decomp. gp assoc. to the diagonal $\subseteq X \times_k X$. Then it holds that

$$\sigma(\mathbb{D}) \;=\; \pi \cdot \mathbb{D} \cdot \pi^{-1} \;\; (\pi \in \Pi_2).$$

Thus, since

$$\begin{tikzcd} \mathbb{D} \arrow[r, hook] \arrow[d, two heads] & \Pi_2 \arrow[d, two heads, "{(p_1, p_2)}"] \\ \{(a,a)\} \arrow[r, hook] & \Pi_1 \times \Pi_1 \end{tikzcd}$$

we conclude that

$$\sigma_2 = \mathrm{Inn}(g_2) \;\; (g_2 \in \Pi_1).$$

Let $g \in \Pi_2$ s.t. $p_1(g) = g_1$ and $p_2(g) = g_2$.

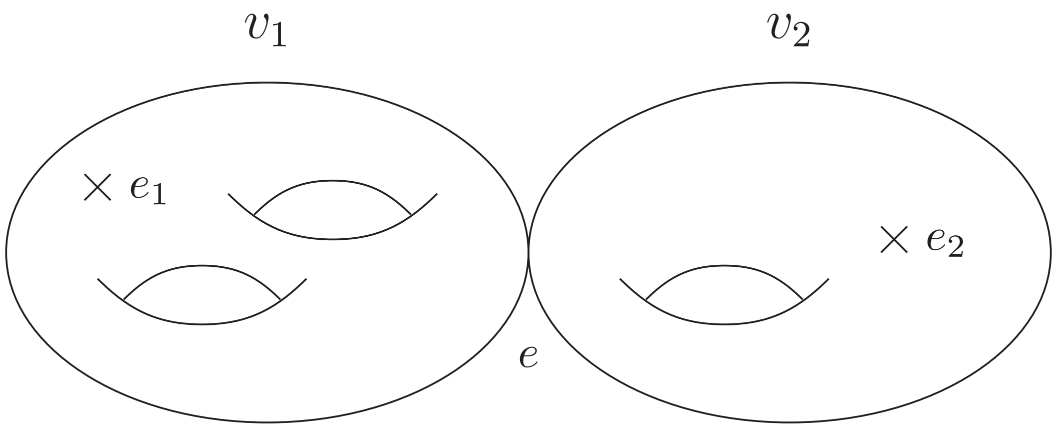
$$\rightsquigarrow \quad \mathrm{Inn}(g)^{-1} \circ \sigma \;\in\; \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2) \;\overset{\sim}{\leftarrow}\; \Xi$$

$$\rightsquigarrow \quad \sigma \in \mathrm{Inn}(\Pi_2)$$

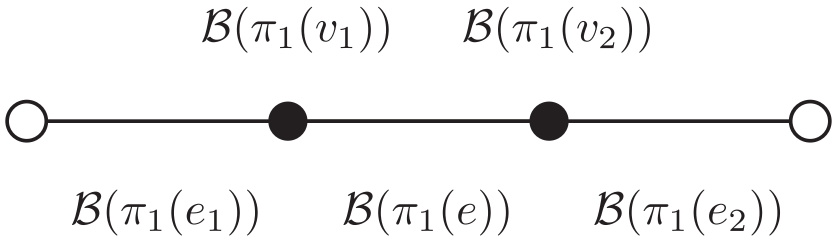
In the following, we consider the proof of Prop 3.

$\boxed{\S 2}$

A pointed stable curve / k



A semi-graph of anabelioids of PSC-type



— where

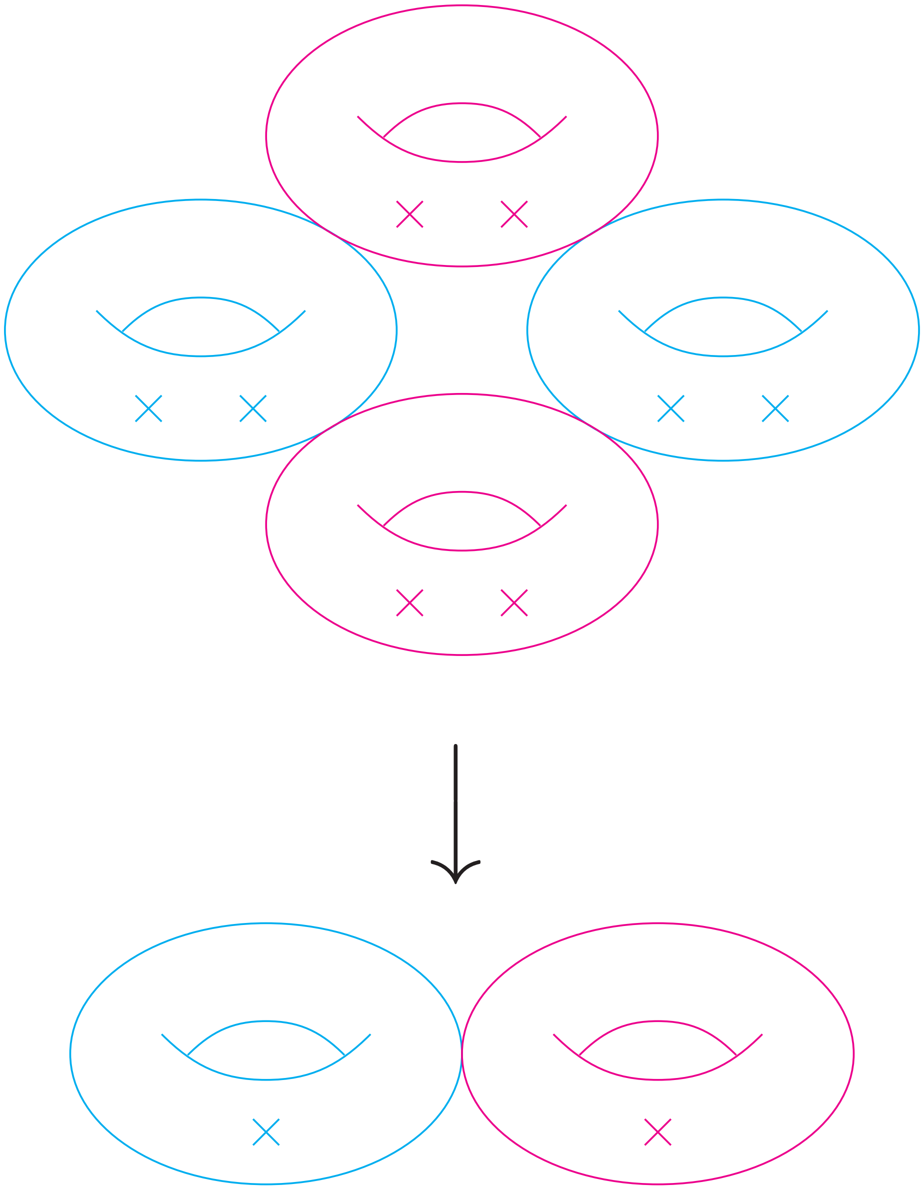
- $*$ irreducible component $\leftrightarrow$ vertex
- $*$ node $\leftrightarrow$ closed edge
- $*$ cusp $\leftrightarrow$ open edge
- $*$ $\mathcal{B}(G)$: the connected anabelioid assoc. to G

$\mathcal{G}$: a semi-graph of anabelioids of PSC-type

$\rightsquigarrow$ A natural notion of finite étale coverings of $\mathcal{G}$

$\rightsquigarrow$ One can define the [profinite] fundamental gp $\Pi_{\mathcal{G}}$

Example: (A finite étale covering of degree 4)



Notation:

$\mathcal{G}$: a semi-graph of anabelioids of PSC-type

$\mathbb{G}$: the underlying semi-graph of $\mathcal{G}$

$\mathcal{V}(\mathcal{G})$: the set of vertices of $\mathbb{G}$

$\mathcal{N}(\mathcal{G})$: the set of nodes of $\mathbb{G}$

$\mathcal{C}(\mathcal{G})$: the set of cusps of $\mathbb{G}$

$\mathcal{E}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathcal{N}(\mathcal{G}) \cup \mathcal{C}(\mathcal{G})$

In particular, $z \in \mathcal{V}(\mathcal{G})$ (resp. $\mathcal{N}(\mathcal{G})$; $\mathcal{C}(\mathcal{G})$; $\mathcal{E}(\mathcal{G})$) determines a(n) verticial (resp. nodal; cuspidal; edge-like) subgroup $\Pi_z \subseteq \Pi_{\mathcal{G}}$ [up to $\Pi_{\mathcal{G}}$-conjugacy].

G: a profinite group $\supseteq$ H: a closed subgroup

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot h = h \cdot g \ \text{ for } {}^{\forall} h \in H\}$$

$$\cap$$

$$N_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$

$$\cap$$

$$C_G(H) \stackrel{\mathrm{def}}{=} \left\{ g \in G \;\middle|\; \begin{array}{ccc} & & g \cdot H \cdot g^{-1} \\ & \overset{\text{open}}{\nearrow} & \\ g \cdot H \cdot g^{-1} \cap H & & \\ & \underset{\text{open}}{\searrow} & \\ & & H \end{array} \right\}$$

$\boxed{\text{Prop 4}}$

(i) If $z \in \mathcal{V}(\mathcal{G})$ (resp. $\mathcal{E}(\mathcal{G})$), then Π_z is slim (resp. $\cong \widehat{\mathbb{Z}}$).

(ii) Let $\{z_1, z_2\}$ be a subset of either $\mathcal{V}(\mathcal{G})$ or $\mathcal{E}(\mathcal{G})$. If $\Pi_{z_1} \cap \Pi_{z_2} \overset{\text{open}}{\subseteq} \Pi_{z_2}$, then $z_1 = z_2$.

(iii) If $z \in \mathcal{V}(\mathcal{G}) \cup \mathcal{E}(\mathcal{G})$, then $C_{\Pi_\mathcal{G}}(\Pi_z) = \Pi_z$.

(iv) Let $z \in \mathcal{E}(\mathcal{G})$. Then $\Pi_z \subseteq \Pi_\mathcal{G}$ is cuspidal (resp. nodal) $\Leftrightarrow$ Π_z is contained in precisely one (resp. two) verticial subgroup(s). ■

$\boxed{\text{Definition}}$

$\mathcal{G}$, $\mathcal{H}$: semi-graphs of anabelioids of PSC-type

$\alpha : \Pi_\mathcal{G} \xrightarrow{\sim} \Pi_\mathcal{H}$: an isomorphism of profinite groups

(i) α is graphic $\Leftrightarrow$ α arises from an isom $\mathcal{G} \xrightarrow{\sim} \mathcal{H}$

(ii) α is group-theoretically verticial (resp. nodal; cuspidal; edge-like) $\Leftrightarrow$ α maps each verticial (resp. nodal; cusp'l; edge-like) subgp $\subseteq \Pi_\mathcal{G}$ onto a verticial (resp. nodal; cusp'l; edge-like) subgp $\subseteq \Pi_\mathcal{H}$, and, every verticial (resp. nodal; cusp'l; edge-like) subgp $\subseteq \Pi_\mathcal{H}$ arises in this fashion.

$\boxed{\text{Prop 5}}$

$\alpha : \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$: an isomorphism of profinite groups

Then α is graphic $\Leftrightarrow$ α is group-theoretically verticial and group-theoretically edge-like.

Moreover, in this case, α arises a unique isom $\mathcal{G} \xrightarrow{\sim} \mathcal{H}$. $\blacksquare$

(Proof) Prop 5 follows from Prop 4, (ii), (iii).

<u>Observe</u>: $\Pi_{\mathcal{G}}$ is topologically finitely generated

$\rightsquigarrow$ A profinite topology on $\mathrm{Out}(\Pi_{\mathcal{G}})$

Since the natural hom. $\mathrm{Aut}(\mathcal{G}) \to \mathrm{Out}(\Pi_{\mathcal{G}})$ is an injection with closed image [cf. Prop 5],

$\rightsquigarrow$ A profinite topology on $\mathrm{Aut}(\mathcal{G})$

$\boxed{\text{Definition}}$

I: a profinite group

We refer to a continuous hom.

$$I \to \mathrm{Aut}(\mathcal{G}) \ [\hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})]$$

as an outer representation of PSC-type.

$\boxed{\text{Thm 6}}$ (A comb. ver. of the Grothendieck Conj.)

$\mathcal{G}$, $\mathcal{H}$: semi-graphs of anabelioids of PSC-type

$\rho_I : I \to \mathrm{Aut}(\mathcal{G})$, $\rho_J : J \to \mathrm{Aut}(\mathcal{H})$:
outer representations of PSC-type

$\alpha : \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$: an isomorphism of profinite groups which fits into a commutative diagram

$$\begin{array}{ccccc} I & \xrightarrow{\rho_I} & \mathrm{Aut}(\mathcal{G}) & \hookrightarrow & \mathrm{Out}(\Pi_{\mathcal{G}}) \\ \downarrow \wr & & & & \downarrow \wr \ \mathrm{Out}(\alpha) \\ J & \xrightarrow{\rho_J} & \mathrm{Aut}(\mathcal{H}) & \hookrightarrow & \mathrm{Out}(\Pi_{\mathcal{H}}) \end{array}$$

— where $I \xrightarrow{\sim} J$ is an isomorphism. Suppose that

(i) ρ_I, ρ_J are of NN-type [cf. §4].

(ii) $\mathcal{C}(\mathcal{G}) \neq \emptyset$ and α is group-theoretically cusp'l.

Then α is graphic [cf. Prop 5]. ■

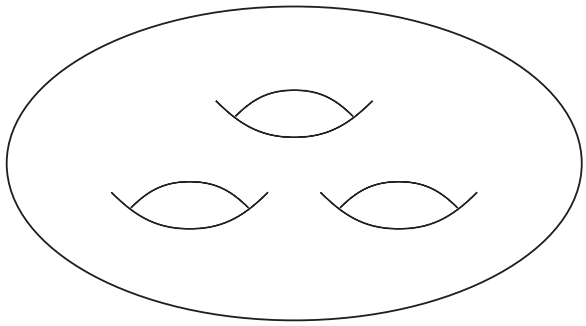
$\boxed{\S 3}$

(Proof of Prop 3)

<u>Remark</u>: At last year's seminar, we have already discussed the affine case of Prop 3!

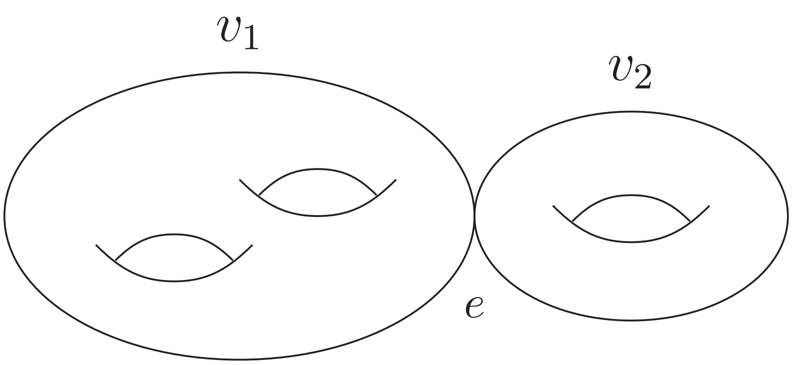
For simplicity, let

$X =$



To verify Prop 3, we may replace $X \to \mathrm{Spec}(k)$ by a stable log curve

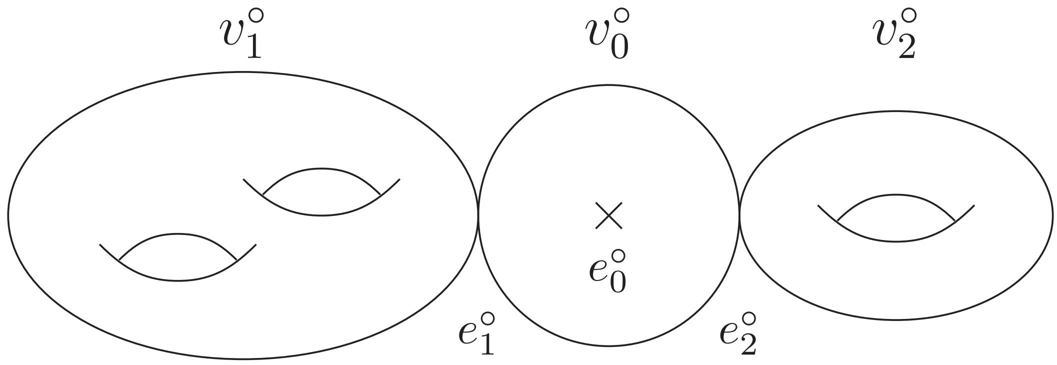
$Z^{\log} =$



over a log point $S^{\log}$. [cf. the deformation theory of stable log curves; the specialization isomorphism $\Pi_n = \pi_1(X_n) \xrightarrow{\sim} \mathrm{Ker}(\pi_1^{\log}(Z_n^{\log}) \twoheadrightarrow \pi_1^{\log}(S^{\log}))$]

$Z^{\log}$ $\rightsquigarrow$ a semi-graph of anabelioids of PSC-type $\mathcal{G}$

<u>Observe</u>: The fiber $(Z_2^{\log})_e$ of $\mathrm{pr}_1 : Z_2^{\log} \to Z^{\log}$ at e is



$\rightsquigarrow$ a semi-graph of anabelioids of PSC-type $\mathcal{G}_{/e}$

In particular, we have

$$\begin{tikzcd}
1 \arrow[r] & \Pi_{2/1} \arrow[r] & \Pi_2 \arrow[r, "p_1"] & \Pi_1 \arrow[r] & 1 \\
 & \Pi_{\mathcal{G}_{/e}} \arrow[u, "\wr"'] & & \Pi_{\mathcal{G}} \arrow[u, "\wr"'] &
\end{tikzcd}$$

— where $\Pi_{2/1} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_2 \twoheadrightarrow \Pi_1)$; the horizontal seq. is exact.

$\rightsquigarrow$ an outer rep'n $\rho : \Pi_1 \to \mathrm{Out}(\Pi_{2/1})$

$Y \subseteq Z$: the irreducible component corr. to v_1

$Y^{\log}$: the smooth log curve over $S^{\log}$ determined by the hyperbolic curve $Y \setminus \{e\}$

$\Pi_n^{\mathrm{sub}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\pi_1(Y_n^{\log}) \twoheadrightarrow \pi_1(S^{\log}))$

$\Pi_{2/1}^{\mathrm{sub}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_2^{\mathrm{sub}} \twoheadrightarrow \Pi_1^{\mathrm{sub}})$

Note: The natural closed imm. $Y \hookrightarrow Z$ induces a commutative diagram

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "\mathrm{pr}_1"] & Z_2 \arrow[d, "\mathrm{pr}_1"] \\ Y \arrow[r] & Z \end{tikzcd}$$

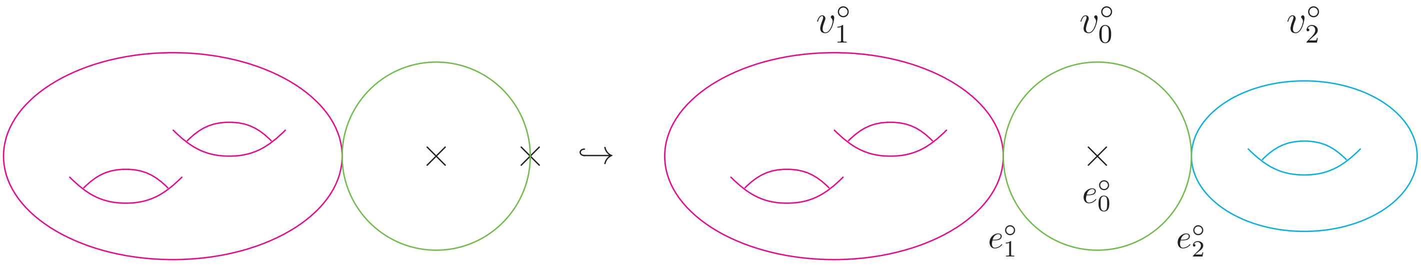
This diagram induces a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \Pi_{2/1}^{\mathrm{sub}} \arrow[r] \arrow[d] & \Pi_2^{\mathrm{sub}} \arrow[r] \arrow[d] & \Pi_1^{\mathrm{sub}} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \Pi_{2/1} \arrow[r] & \Pi_2 \arrow[r, "p_1"] & \Pi_1 \arrow[r] & 1 \end{tikzcd}$$

— where the horizontal seq. is exact; the vertical arrows are injective.

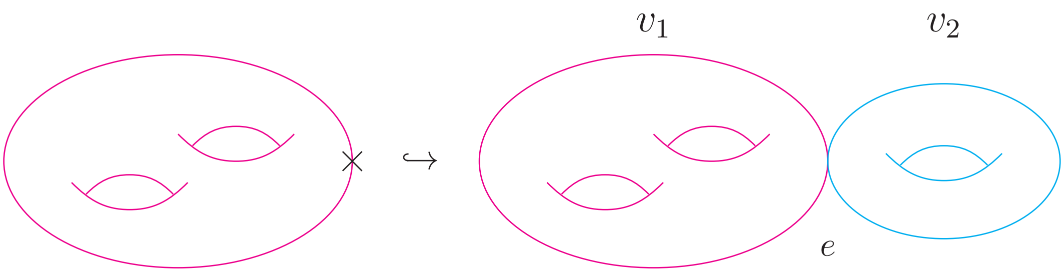
$\rightsquigarrow$ an outer rep'n $\rho^{\mathrm{sub}} : \Pi_1^{\mathrm{sub}} \to \mathrm{Out}(\Pi_{2/1}^{\mathrm{sub}})$

$\underline{\Pi_{2/1}^{\mathrm{sub}} \hookrightarrow \Pi_{2/1}}$



$(Y_2^{\log})_e$ $\qquad$ $(Z_2^{\log})_e$

$\underline{\Pi_{1}^{\mathrm{sub}} \hookrightarrow \Pi_{1}}$



$Y^{\log}$ $\qquad$ $Z^{\log}$

Let $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$.

$\Pi_e \subseteq \Pi_1$: an edge-like subgp assoc. to e

$\rightsquigarrow$ We have a comm. diag.

$$\begin{array}{ccccc} \Pi_e & \hookrightarrow & \Pi_1 & \xrightarrow{\rho} & \mathrm{Out}(\Pi_{2/1}) \\ \| & & \| & & \downarrow \wr\ \mathrm{Out}(\alpha|_{\Pi_{2/1}}) \\ \Pi_e & \hookrightarrow & \Pi_1 & \xrightarrow{\rho} & \mathrm{Out}(\Pi_{2/1}) \end{array}$$

<u>Note</u>: Since the composite $\Pi_e \hookrightarrow \Pi_1 \to \mathrm{Out}(\Pi_{2/1})$ factors as the composite of

$$\Pi_e \ \to\ \mathrm{Aut}(\mathcal{G}_{/e}) \ \text{ with } \ \mathrm{Aut}(\mathcal{G}_{/e}) \ \hookrightarrow\ \mathrm{Out}(\Pi_{2/1})$$

— where the resulting outer rep'n is of NN-type — it follows from Thm 6 that $\alpha|_{\Pi_{2/1}}$ is graphic.

Moreover, since

$$\alpha \circlearrowright \Pi_2 \xrightarrow{p_2} \!\!\!\!\rightarrow \Pi_1 \circlearrowleft \alpha_2 = \mathrm{id}$$

$\alpha|_{\Pi_{2/1}}$ induces identity automorphism on "$\mathbb{G}_{/e}$" [cf. Prop 4, (i), (ii)].

Fix an edge-like subgroup $\Pi_{e_1^\circ} \subseteq \Pi_{2/1}$ assoc. to e_1°.

<u>Claim</u>: ${}^\exists \gamma \in \Xi$ s.t.

$$\alpha(\Pi_{e_1^\circ}) \;=\; \gamma \cdot \Pi_{e_1^\circ} \cdot \gamma^{-1}.$$

(Proof of Claim)

By the above <u>Note</u>, there exists $\gamma' \in \Pi_{2/1}$ s.t.

$$\alpha(\Pi_{e_1^\circ}) \;=\; \gamma' \cdot \Pi_{e_1^\circ} \cdot \gamma'^{-1}.$$

Thus, we have

$$p_2(\Pi_{e_1^\circ}) \;=\; p_2(\gamma') \cdot p_2(\Pi_{e_1^\circ}) \cdot \; p_2(\gamma')^{-1}.$$

By Prop4, (iii), we conclude that

$$p_2(\gamma') \;\in\; p_2(\Pi_{e_1^\circ}).$$

In particular, by multiplying γ' by a suitable $\in \Pi_{e_1^\circ}$, we obtain an element $\gamma \in \Xi$, as desired.

In light of Claim, to verify that $\alpha \in \Xi$, we may assume that

$$(1) \quad \alpha(\Pi_{e_1^\circ}) \;=\; \Pi_{e_1^\circ}.$$

$\Pi_{v_1^\circ}$, $\Pi_{v_0^\circ} \subseteq \Pi_{2/1}$: the unique verticial subgps assoc. to v_1°, v_0° that contain $\Pi_{e_1^\circ}$ [cf. Prop 4, (iv)]

$\Pi_{2/1}^{\mathrm{sub}} \subseteq \Pi_{2/1}$: the unique $\Pi_{2/1}$-conj. of the image of "$\Pi_{2/1}^{\mathrm{sub}} \hookrightarrow \Pi_{2/1}$" that contains and is topologically generated by $\Pi_{v_1^\circ}$, $\Pi_{v_0^\circ}$

$\rightsquigarrow$ By (1) and the graphicity of $\alpha|_{\Pi_{2/1}}$, we conclude:

(2) $\alpha(\Pi_{v_1^\circ}) = (\Pi_{v_1^\circ})$;

(3) $\alpha(\Pi_{v_0^\circ}) = (\Pi_{v_0^\circ})$;

(4) $\alpha(\Pi_{2/1}^{\mathrm{sub}}) = (\Pi_{2/1}^{\mathrm{sub}})$.

<u>Observe</u>: Since $C_{\Pi_{2/1}}(\Pi_{2/1}^{\mathrm{sub}}) = \Pi_{2/1}^{\mathrm{sub}}$ [cf. Prop 4, (iii)], the diagram

$$\begin{tikzcd}
\Pi_1^{\mathrm{sub}} \arrow[r, "\rho^{\mathrm{sub}}"] \arrow[d, equal] & \mathrm{Out}(\Pi_{2/1}^{\mathrm{sub}}) \arrow[d, "\wr", "\mathrm{Out}(\alpha|_{\Pi_{2/1}^{\mathrm{sub}}})"'] \\
\Pi_1^{\mathrm{sub}} \arrow[r, "\rho^{\mathrm{sub}}"] & \mathrm{Out}(\Pi_{2/1}^{\mathrm{sub}})
\end{tikzcd}$$

[cf. (4)] commutes. Thus, $\alpha|_{\Pi_{2/1}^{\mathrm{sub}}}$ arises from an $\alpha^{\mathrm{sub}} \in \mathrm{Aut}(\Pi_2^{\mathrm{sub}})$ [cf. the slimness of $\Pi_{2/1}^{\mathrm{sub}}$].

Moreover, it follows from the construction that

$$\alpha^{\mathrm{sub}} \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2^{\mathrm{sub}}).$$

On the other hand, by the affine case of Prop 3 [cf. Remark at the beginning of the proof], we have

$$\mathrm{Aut}^{\mathrm{IFC}}(\Pi_2^{\mathrm{sub}}) \overset{\sim}{\leftarrow} \Xi^{\mathrm{sub}} \overset{\mathrm{def}}{=} \Xi \cap \Pi_2^{\mathrm{sub}}.$$

$\rightsquigarrow$ $\alpha|_{\Pi_{2/1}^{\mathrm{sub}}}$ is a Ξ-inner automorphism!

Thus, to verify that $\alpha \in \Xi$, we may assume that

$$(5) \quad \alpha|_{\Pi_{2/1}^{\mathrm{sub}}} = \mathrm{id}.$$

$$\rightsquigarrow \; (6) \quad \alpha|_{\Pi_{v_0^\circ}} = \mathrm{id} \quad [\text{cf. } (3)]$$

$\Pi_{e_2^\circ} \subseteq \Pi_{2/1}$: an edge-like subgp assoc. to e_2° which is contained in $\Pi_{v_0^\circ}$

$\Pi_{v_2^\circ} \subseteq \Pi_{2/1}$: the unique verticial subgp assoc. to v_2 that contains $\Pi_{e_2^\circ}$ [cf. Prop 4, (iv)]

$$\rightsquigarrow \; (7) \quad \alpha(\Pi_{e_2^\circ}) = \Pi_{e_2^\circ}$$

By (7) and the graphicity of $\alpha|_{\Pi_{2/1}}$, we conclude:

$$(8) \quad \alpha(\Pi_{v_2^\circ}) = \Pi_{v_2^\circ}.$$

By (8), we obtain a comm. diag.

$$\begin{tikzcd}
\Pi_{v_2^\circ} \arrow[r, "\sim", "\alpha|_{\Pi_{v_2^\circ}}"'] \arrow[d, "\wr"] & \Pi_{v_2^\circ} \arrow[d, "\wr"] \\
p_2(\Pi_{v_2^\circ}) \arrow[r, equal] & p_2(\Pi_{v_2^\circ})
\end{tikzcd}$$

Hence, we have

$$(9) \quad \alpha|_{\Pi_{v_2^\circ}} = \mathrm{id}.$$

Since $\Pi_{2/1}$ is topologically generated by $\Pi_{2/1}^{\mathrm{sub}}$ and $\Pi_{v_2^\circ}$, it follows from (5), (9) that

$$(10) \quad \alpha|_{\Pi_{2/1}} = \mathrm{id}.$$

Finally, it follows from (10) and the assumption

$$\alpha \circlearrowright \Pi_2 \xrightarrow{p_1} \Pi_1 \circlearrowleft \alpha_1 = \mathrm{id}$$

that $\alpha = \mathrm{id}$ [cf. the slimness of $\Pi_{2/1}$].

$\boxed{\S 4}$

Fix a universal covering $\widetilde{\mathcal{G}} \to \mathcal{G}$.

$\mathcal{V}(\widetilde{\mathcal{G}}) \stackrel{\text{def}}{=} \varprojlim \mathcal{V}(\mathcal{G}')$

$\mathcal{N}(\widetilde{\mathcal{G}}) \stackrel{\text{def}}{=} \varprojlim \mathcal{N}(\mathcal{G}')$

$\mathcal{C}(\widetilde{\mathcal{G}}) \stackrel{\text{def}}{=} \varprojlim \mathcal{C}(\mathcal{G}')$

$\mathcal{E}(\widetilde{\mathcal{G}}) \stackrel{\text{def}}{=} \mathcal{N}(\widetilde{\mathcal{G}}) \cup \mathcal{C}(\widetilde{\mathcal{G}})$

— where the proj. limits are over all conn. fin. étale subcoverings $\mathcal{G}' \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$

Let $\square \in \{\mathcal{V}, \mathcal{E}\}$.

If $\tilde{z} \in \square(\widetilde{\mathcal{G}})$, then we write $\tilde{z}(\mathcal{G}')$ for the image of $\tilde{z}$ via the natural map $\square(\widetilde{\mathcal{G}}) \to \square(\mathcal{G}')$.

$\Pi_{\tilde{z}} \subseteq \Pi_{\mathcal{G}}$: a unique $\square$-like subgroup assoc. to $\tilde{z}(\mathcal{G})$ s.t. for every conn. fin. étale subcovering $\mathcal{G}' \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$, the subgroup $\Pi_{\tilde{z}} \cap \Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}'}$ is a $\square$-like subgroup assoc. to $\tilde{z}(\mathcal{G}')$

$\rho_I : I \to \mathrm{Aut}(\mathcal{G})$: an outer rep'n of PSC-type

$\Pi_I \overset{\mathrm{def}}{=} \Pi_{\mathcal{G}} \overset{\mathrm{out}}{\rtimes} I$: the profinite group obtained by pulling back the exact sequence

$$1 \longrightarrow \Pi_{\mathcal{G}} \xrightarrow{\text{conj.}} \mathrm{Aut}(\Pi_{\mathcal{G}}) \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}}) \longrightarrow 1$$

by the composite $I \overset{\rho_I}{\to} \mathrm{Aut}(\mathcal{G}) \hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$.

$\rightsquigarrow$ we have a comm. diag.

$$\begin{tikzcd}
1 \arrow[r] & \Pi_{\mathcal{G}} \arrow[r] \arrow[d, equal] & \Pi_I \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, "\rho_I"] & 1 \\
1 \arrow[r] & \Pi_{\mathcal{G}} \arrow[r] & \mathrm{Aut}(\Pi_{\mathcal{G}}) \arrow[r] & \mathrm{Out}(\Pi_{\mathcal{G}}) \arrow[r] & 1
\end{tikzcd}$$

— where the horizontal seq. are exact.

Definition

(i) If $\tilde{z} \in \mathcal{V}(\widetilde{\mathcal{G}})$ or $\mathcal{N}(\widetilde{\mathcal{G}})$, then we shall write

$$I_{\tilde{z}} \overset{\mathrm{def}}{=} Z_{\Pi_I}(\Pi_{\tilde{z}}) \ \subseteq \ D_{\tilde{z}} \overset{\mathrm{def}}{=} N_{\Pi_I}(\Pi_{\tilde{z}}).$$

(ii) If $\tilde{z} \in \mathcal{C}(\widetilde{\mathcal{G}})$, then we shall write

$$I_{\tilde{z}} \overset{\mathrm{def}}{=} \Pi_{\tilde{z}} \ \subseteq \ D_{\tilde{z}} \overset{\mathrm{def}}{=} N_{\Pi_I}(\Pi_{\tilde{z}}).$$

$\boxed{\text{Lem 7}}$

(i) Let $\tilde{v} \in \mathcal{V}(\widetilde{\mathcal{G}})$. Then the composite $I_{\tilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$ is injective.

(ii) Let $\tilde{e} \in \mathcal{N}(\widetilde{\mathcal{G}})$ that abuts to $\tilde{v} \in \mathcal{V}(\widetilde{\mathcal{G}})$. Then it holds that $I_{\tilde{v}} \subseteq I_{\tilde{e}}$. $\blacksquare$

(Proof) (i) follows from Prop 4, (i), (iii). (ii) is easy.

$\boxed{\text{Definition}}$

$\rho_I : I \to \mathrm{Aut}(\mathcal{G})$: an outer rep'n of PSC-type

ρ_I is of NN-type (resp. SNN-type) $\Leftrightarrow$

(1) $I \cong \widehat{\mathbb{Z}}$.

(2) For every $\tilde{v} \in \mathcal{V}(\widetilde{\mathcal{G}})$, the image of the composite $I_{\tilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$ is open (resp. I) [cf. Lem 7, (i)].

(3) For every $\tilde{e} \in \mathcal{N}(\widetilde{\mathcal{G}})$, the natural inclusions $I_{\tilde{v}_1}, I_{\tilde{v}_2} \subseteq I_{\tilde{e}}$ — where $\tilde{e}$ abuts to $\tilde{v}_1$, $\tilde{v}_2 \in \mathcal{V}(\widetilde{\mathcal{G}})$ [cf. Lem 7, (ii)] — induces an open injection $I_{\tilde{v}_1} \times I_{\tilde{v}_2} \hookrightarrow I_{\tilde{e}}$.

$\boxed{\text{Lem 8}}$ $\rho_I : I \to \mathrm{Aut}(\mathcal{G})$: of SNN-type

(i) Let $\tilde{v} \in \mathcal{V}(\widetilde{\mathcal{G}})$. Then $D_{\tilde{v}} = \Pi_{\tilde{v}} \times I_{\tilde{v}}$.

(ii) Let $\tilde{v} \in \mathcal{V}(\widetilde{\mathcal{G}})$; $\tilde{e} \in \mathcal{E}(\widetilde{\mathcal{G}})$ an element that abuts to $\tilde{v}$. Then $D_{\tilde{e}} = \Pi_{\tilde{e}} \times I_{\tilde{v}}$.

(iii) $\Pi_{\tilde{v}} = Z_{\Pi_I}(I_{\tilde{v}}) \cap \Pi_{\mathcal{G}} = N_{\Pi_I}(I_{\tilde{v}}) \cap \Pi_{\mathcal{G}}$. ■

<u>Notation</u>:

(i) Let v, $w \in \mathcal{V}(\mathcal{G})$. We shall write $\delta(v, w) \leq n$ if the following conditions are satisfied:

* If $n = 0$, then $v = w$.
* If $n \geq 1$, then there exist

$$\{e_1, \ldots, e_n\} \subseteq \mathcal{N}(\mathcal{G}); \ \{v_0, \ldots, v_n\} \subseteq \mathcal{V}(\mathcal{G})$$

— where $v_0 = v$; $v_n = w$; e_i abuts to v_{i-1}, v_i.

We shall write $\delta(v, w) = n$ if $\delta(v, w) \leq n$ and $\delta(v, w) \not\leq n - 1$.

(ii) Let $\tilde{v}$, $\tilde{w} \in \mathcal{V}(\widetilde{\mathcal{G}})$. We shall write

$$\delta(\tilde{v}, \tilde{w}) \overset{\text{def}}{=} \sup_{\mathcal{G}'}\{\delta(\tilde{v}(\mathcal{G}'), \tilde{w}(\mathcal{G}'))\}.$$

$\boxed{\text{Lem 9}}$

$\rho_I : I \to \mathrm{Aut}(\mathcal{G})$: an outer rep'n of PSC-type

Let $\tilde{v}_1$, $\tilde{v}_2 \in \mathcal{V}(\widetilde{\mathcal{G}})$.

Consider the following 8 conditions:

(1) $\delta(\tilde{v}_1, \tilde{v}_2) = 0$.

(2) $\delta(\tilde{v}_1, \tilde{v}_2) = 1$.

(3) $\delta(\tilde{v}_1, \tilde{v}_2) = 2$.

(4) $\delta(\tilde{v}_1, \tilde{v}_2) \geq 3$.

(1′) $D_{\tilde{v}_1} = D_{\tilde{v}_2}$.

(2′) $D_{\tilde{v}_1} \neq D_{\tilde{v}_2}$; $D_{\tilde{v}_1} \cap D_{\tilde{v}_2} \cap \Pi_{\mathcal{G}} \neq \{1\}$.

(3′) $D_{\tilde{v}_1} \cap D_{\tilde{v}_2} \neq \{1\}$; $D_{\tilde{v}_1} \cap D_{\tilde{v}_2} \cap \Pi_{\mathcal{G}} = \{1\}$.

(4′) $D_{\tilde{v}_1} \cap D_{\tilde{v}_2} = \{1\}$.

Then we have equivalences

(1) $\Leftrightarrow$ (1′); (2) $\Leftrightarrow$ (2′); (3) $\Leftrightarrow$ (3′); (4) $\Leftrightarrow$ (4′).

Moreover, suppose that ρ_I is of SNN-type.

Then if (3′) is satisfied, then there exists a unique $\tilde{v}_3 \in \mathcal{V}(\widetilde{\mathcal{G}})$ s.t.

$$\delta(\tilde{v}_1, \tilde{v}_3) = \delta(\tilde{v}_2, \tilde{v}_3) = 1 \ \text{ and } \ D_{\tilde{v}_1} \cap D_{\tilde{v}_2} = I_{\tilde{v}_3}. \ \blacksquare$$

(Proof of Thm 6)

To verify Thm 6, by replacing Π_I by an open subgroup $\subseteq \Pi_I$, we may assume that $\mathcal{G}$, $\mathcal{H}$ are sturdy, and that ρ_I, ρ_J are of SNN-type [cf. Prop 4, (iii)].

$*$ sturdy $\cdots$ Every irr. component of the pointed stable curve that gives rise to $\mathcal{G}$ satisfies the following:

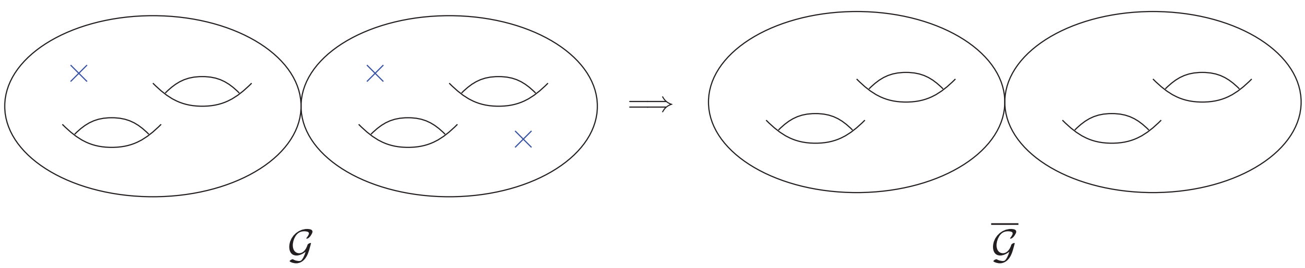
The genus of the normalization is ≥ 2.

Thus, to verify Thm 6, it suffices to verify

Claim A:

$\overline{\mathcal{G}}$, $\overline{\mathcal{H}}$: the compactifications of $\mathcal{G}$, $\mathcal{H}$, respectively
Then the isom. $\overline{\alpha} : \Pi_{\overline{\mathcal{G}}} \overset{\sim}{\to} \Pi_{\overline{\mathcal{H}}}$ which is induced by α is graphic. ■

[We apply Claim A to various conn. fin. ét. coverings!]



$\boxed{\text{Lem 10}}$

Gp-theoretically verticial $\Rightarrow$ Gp-theoretically nodal. ■

It follows from Prop 5, Lem 10 that Claim A $\Leftarrow$

Claim B:

$\overline{\alpha} : \Pi_{\overline{\mathcal{G}}} \xrightarrow{\sim} \Pi_{\overline{\mathcal{H}}}$ is gp-theoretically verticial. ■

(Proof of Claim B) First, let us prove that:

There exists a verticial subgroup $\nabla \subseteq \Pi_{\overline{\mathcal{G}}}$ s.t. $\overline{\alpha}(\nabla)$ is a verticial subgroup $\subseteq \Pi_{\overline{\mathcal{H}}}$.

Write $I \to \mathrm{Out}(\Pi_{\overline{\mathcal{G}}})$ (resp. $J \to \mathrm{Out}(\Pi_{\overline{\mathcal{H}}})$) for the outer rep'n of PSC-type determined by ρ_I (resp. ρ_J) and

$$\overline{\Pi}_I \overset{\text{def}}{=} \Pi_{\overline{\mathcal{G}}} \overset{\text{out}}{\rtimes} I, \quad \overline{\Pi}_J \overset{\text{def}}{=} \Pi_{\overline{\mathcal{H}}} \overset{\text{out}}{\rtimes} J$$

$\rightsquigarrow$ $\alpha : \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ [and $\overline{\alpha}$] induces a comm. diag.

$$(\star) \quad \begin{array}{ccc} \Pi_I = \Pi_{\mathcal{G}} \overset{\text{out}}{\rtimes} I & \underset{\beta}{\xrightarrow{\sim}} & \Pi_{\mathcal{H}} \overset{\text{out}}{\rtimes} J = \Pi_J \\ \downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow \\ \overline{\Pi}_I = \Pi_{\overline{\mathcal{G}}} \overset{\text{out}}{\rtimes} I & \underset{\overline{\beta}}{\xrightarrow{\sim}} & \Pi_{\overline{\mathcal{H}}} \overset{\text{out}}{\rtimes} J = \overline{\Pi}_J \end{array}$$

— where the vertical arrows are the surj. induced by $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\overline{\mathcal{G}}}$, $\Pi_{\mathcal{H}} \twoheadrightarrow \Pi_{\overline{\mathcal{H}}}$.

By assumption (ii), $^{\exists}e_{\mathcal{G}} \in \mathcal{C}(\mathcal{G})$, $^{\exists}e_{\mathcal{H}} \in \mathcal{C}(\mathcal{H})$ s.t.

$$\beta(D_{e_{\mathcal{G}}}) = D_{e_{\mathcal{H}}}.$$

Write $v_{\mathcal{G}} \in \mathcal{V}(\mathcal{G})$ (resp. $v_{\mathcal{H}} \in \mathcal{V}(\mathcal{H})$) for the vertex to which $e_{\mathcal{G}}$ (resp. $e_{\mathcal{H}}$) abuts.

By Lem 8, (ii), the diag. $(\star)$ induces a diag.

$$\begin{tikzcd} D_{e_{\mathcal{G}}} \arrow[r, "\sim", "\beta"'] \arrow[d, two heads] & D_{e_{\mathcal{H}}} \arrow[d, two heads] \\ I_{v_{\mathcal{G}}} \arrow[r, "\sim", "\overline{\beta}"'] & I_{v_{\mathcal{H}}} \end{tikzcd} \quad \rightsquigarrow \quad \overline{\beta}(I_{v_{\mathcal{G}}}) = I_{v_{\mathcal{H}}} \ !$$

Thus, we conclude from Lem 8, (iii), that

$$\begin{aligned} \overline{\alpha}(\Pi_{v_{\mathcal{G}}}) &= \overline{\beta}(N_{\overline{\Pi}_I}(I_{v_{\mathcal{G}}}) \ \cap \ \Pi_{\overline{\mathcal{G}}}) \\ &= N_{\overline{\Pi}_I}(\overline{\beta}(I_{v_{\mathcal{G}}})) \ \cap \ \Pi_{\overline{\mathcal{H}}} \\ &= N_{\overline{\Pi}_I}(I_{v_{\mathcal{H}}}) \ \cap \ \Pi_{\overline{\mathcal{H}}} \\ &= \Pi_{v_{\mathcal{H}}}. \quad \rightsquigarrow \quad \nabla \overset{\text{def}}{=} \Pi_{v_{\mathcal{G}}} \end{aligned}$$

Therefore, to verify Claim B, it suffices to show that:

Let $\tilde{v}_1, \tilde{v}_2 \in \mathcal{V}(\tilde{\overline{\mathcal{G}}})$ s.t. $\delta(\tilde{v}_1(\overline{\mathcal{G}}), \tilde{v}_2(\overline{\mathcal{G}})) \leq 1$. Then if $\overline{\alpha}(\Pi_{\tilde{v}_1})$ is verticial, then $\overline{\alpha}(\Pi_{\tilde{v}_2})$ is verticial.

If $\tilde{v}_1(\overline{\mathcal{G}}) = \tilde{v}_2(\overline{\mathcal{G}})$, then it is immediate. Suppose that $\tilde{v}_1(\overline{\mathcal{G}}) \neq \tilde{v}_2(\overline{\mathcal{G}})$ and that $\overline{\alpha}(\Pi_{\tilde{v}_1})$ is verticial.

<u>Observe</u>: There exist $\tilde{w}_1, \tilde{u}_1, \tilde{w}_2 \in \mathcal{V}(\widetilde{\overline{\mathcal{G}}})$ s.t.

(a) $\tilde{v}_1(\overline{\mathcal{G}}) = \tilde{w}_1(\overline{\mathcal{G}}) = \tilde{u}_1(\overline{\mathcal{G}})$; $\tilde{v}_2(\overline{\mathcal{G}}) = \tilde{w}_2(\overline{\mathcal{G}})$.

(b) $\delta(\tilde{w}_1, \tilde{u}_1) = 2$.

(c) $\delta(\tilde{w}_2, \tilde{w}_1) = \delta(\tilde{w}_2, \tilde{u}_1) = 1$. [cf. the next page]

(a) $\rightsquigarrow$ There exist $\tilde{w}'_1, \tilde{u}'_1 \in \mathcal{V}(\widetilde{\overline{\mathcal{H}}})$ s.t.

$$\overline{\beta}(D_{\tilde{w}_1}) = D_{\tilde{w}'_1}, \quad \overline{\beta}(D_{\tilde{u}_1}) = D_{\tilde{u}'_1}.$$

(b), (c), Lem 9 $\rightsquigarrow$ $D_{\tilde{w}_1} \cap D_{\tilde{u}_1} = I_{\tilde{w}_2}$

$\rightsquigarrow$ $D_{\tilde{w}'_1} \cap D_{\tilde{u}'_1} \neq \{1\}$, $D_{\tilde{w}'_1} \cap D_{\tilde{u}'_1} \cap \Pi_{\overline{\mathcal{H}}} = \{1\}$

$\rightsquigarrow$ ${}^{\exists}\tilde{w}'_2 \in \mathcal{V}(\widetilde{\overline{\mathcal{H}}})$ s.t. $D_{\tilde{w}'_1} \cap D_{\tilde{u}'_1} = I_{\tilde{w}'_2}$ [cf. Lem 9]

$\rightsquigarrow$ $\overline{\beta}(I_{\tilde{w}_2}) = I_{\tilde{w}'_2}$

Thus, it follows from Lem 8, (iii), that

$$\overline{\alpha}(\Pi_{\tilde{w}_2}) = \Pi_{\tilde{w}'_2}.$$

$\rightsquigarrow$ We conclude from (a) that $\overline{\alpha}(\Pi_{\tilde{v}_2})$ is verticial !

$\overline{\mathcal{G}}' \to \overline{\mathcal{G}}$: a connected finite étale covering of degree $= 2$